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*Published in:*  
Journal of Applied Physics

*Link to article, DOI:*  
[10.1063/1.1663674](https://doi.org/10.1063/1.1663674)

*Publication date:*  
1974

*Document Version*  
Publisher's PDF, also known as Version of record

[Link back to DTU Orbit](#)

*Citation (APA):*  
Karlsson, P. W. (1974). Uniqueness of time-independent electromagnetic fields. *Journal of Applied Physics*, 45(6), 2795-2795. <https://doi.org/10.1063/1.1663674>

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# Uniqueness of time-independent electromagnetic fields

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(Received 26 December 1973; in final form 26 February 1974)

As a comment on a recent paper by Steele, a more general uniqueness theorem for time-independent fields is mentioned.

Uniqueness theorems for time-independent electromagnetic fields in linear media are found in many textbooks. Recently, Steele<sup>1</sup> proved theorems of this kind for nonlinear isotropic media with positive differential permeability and permittivity. This result, however, is essentially contained in a more general theorem due to Graffi,<sup>2</sup> which will be mentioned here since Ref. 2 is not easily accessible. To facilitate applications, a somewhat extended form of Graffi's theorem will be given.

Consider a vector field  $\mathbf{Y}$  that depends upon another vector field  $\mathbf{X}$ . Let  $R$  denote a simply connected region bounded by closed surfaces  $S_1, S_2, \dots, S_m$ ;  $S_m$  is the external boundary, absent when  $R$  is unbounded, and  $\mathbf{n}$  denotes the outward normal unit vector. We introduce the following assumptions:

- (i)  $\mathbf{X}_1 \neq \mathbf{X}_2$  implies  $\mathbf{Y}(\mathbf{X}_1) \neq \mathbf{Y}(\mathbf{X}_2)$  and  $(\mathbf{X}_1 - \mathbf{X}_2) \cdot (\mathbf{Y}(\mathbf{X}_1) - \mathbf{Y}(\mathbf{X}_2)) > 0$ ;
- (ii) In  $R$ ,  $\nabla \times \mathbf{X}$ , and  $\nabla \cdot \mathbf{Y}$  are prescribed;
- (iii) If  $R$  is unbounded,  $|\mathbf{X}|$  and  $|\mathbf{Y}|$  vanish as  $r^{-2}$  when  $r \rightarrow \infty$ ,  $r$  denotes the distance from some fixed point;
- (iv) For each  $i$  ( $= 1, 2, \dots, m$ ) either  $\mathbf{n} \cdot \mathbf{Y}$  or  $\mathbf{n} \times \mathbf{X}$  on  $S_i$  is prescribed;
- (v) If for  $i < m$ ,  $\mathbf{n} \times \mathbf{X}$  on  $S_i$  is prescribed, then either  $\int_{S_i} \mathbf{n} \cdot \mathbf{Y} dS$  or  $\int_{C_i} \mathbf{X} \cdot d\mathbf{s}$  is prescribed, where  $C_i$  is a curve in  $R$  from a point on  $S_i$  to a fixed point  $Q$  on  $S_m$  if  $R$  is bounded, otherwise to infinity.

Then the vector field  $\mathbf{X}$ , and thus  $\mathbf{Y}$ , is uniquely determined.

To prove this uniqueness theorem we assume (as usual) the existence of two solutions,  $\mathbf{X}_1$  and  $\mathbf{X}_2$ , introduce the difference fields

$$\begin{aligned}\mathbf{X}_d &= \mathbf{X}_1 - \mathbf{X}_2, \\ \mathbf{Y}_d &= \mathbf{Y}_1 - \mathbf{Y}_2 \\ &= \mathbf{Y}(\mathbf{X}_1) - \mathbf{Y}(\mathbf{X}_2),\end{aligned}$$

and consider the integral

$$I = \int_R \mathbf{X}_d \cdot \mathbf{Y}_d dv.$$

It follows from (ii) that  $\nabla \times \mathbf{X} = 0$  in  $R$ ; we can thus write  $\mathbf{X}_d = -\nabla U$  with  $U(P) = \int_P^Q \mathbf{X}_d \cdot d\mathbf{s}$ . Moreover, (ii) implies

$\nabla \cdot \mathbf{Y}_d = 0$ . Hence, by the divergence theorem [and (iii) when applicable] we obtain

$$\begin{aligned}I &= \int_R -(\nabla U) \cdot \mathbf{Y}_d dv \\ &= - \int_R \nabla \cdot (\mathbf{Y}_d U) dv \\ &= - \sum_{i=1}^m I_i,\end{aligned}$$

where

$$I_i = \int_{S_i} \mathbf{n} \cdot (\mathbf{Y}_d U) dS.$$

We now utilize (iv). Either  $\mathbf{n} \cdot \mathbf{Y}_d = 0$ , which yields  $I_i = 0$ , or  $\mathbf{n} \times \mathbf{X}_d = 0$ , which means that  $U$  is a constant  $U_i$  on  $S_i$ , and thus

$$I_i = U_i \int_{S_i} \mathbf{n} \cdot \mathbf{Y}_d dS.$$

Obviously  $U_m = 0$ . For  $i < m$ , we infer from (v) that either  $U_i = 0$  or  $\int_{S_i} \mathbf{n} \cdot \mathbf{Y}_d dS = 0$ . In any case,  $I_i = 0$ .

It now follows that  $I = 0$ . This is, however, because of (i), impossible unless  $\mathbf{X}_d = 0$  throughout  $R$ . Consequently,  $\mathbf{X}$  is uniquely determined.

A similar theorem holds when  $R$  is multiply connected. In that case we must prescribe together with  $\mathbf{n} \cdot \mathbf{Y}$  all independent circulations  $\oint \mathbf{X} \cdot d\mathbf{s}$  associated with the  $S_i$  in question (e.g., one circulation if  $S_i$  is the surface of a toroid). Then  $U$  becomes one valued and the proof just given remains valid.

The applications of the theorem to classical cases are straightforward. For electrostatic fields we can prescribe the charge density in the dielectric part of space and the potential or charge of each conductor; for stationary magnetic fields the current density is prescribable. In either case, condition (i) defines the class of materials to which the theorem is applicable. Clearly, condition (i) is satisfied when  $\mathbf{X}$  and  $\mathbf{Y}$  are parallel and  $\mathbf{Y}$  is an increasing function of  $\mathbf{X}$ . The problem considered by Steele<sup>1</sup> is thus included as a particular case.

<sup>1</sup>C. W. Steele, J. Appl. Phys. **44**, 3790 (1973).

<sup>2</sup>D. Graffi, in *Scritti matematici in onore di Filippo Sibirani* (Zuffi, Bologna, 1956), p. 143.